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Prediction of Time-Correlated Gust Loads Using an Incremental Stochastic Search

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A new algorithm for finding gust inputs that maximize aircraft response loads subject to a constraint that is defined by an existing requirement is described. Such gust inputs have two main applications: the study of time-correlated loads in stress analysis, and the estimation of design loads in design and certification analysis. The algorithm is applicable to nonlinear systems subject to commonly satisfied restrictions. The performance of the algorithm is demonstrated on linear and nonlinear configurations of a realistic model of a commercial wide-body aircraft, and the results are compared with those generated via two existing search methods.

Nomenclature

$\bar{A}_y$ = ratio of the standard deviation of the output to the standard deviation of the input
b_n = base signal at stage n
d_n = energy difference between the sample mean and the base signal
F = function producing a time-history output given a time-history input
f = function producing the maximum value of the time-history output given a time-history input
g = gust input time history
$H(\omega)$ = von Kármán transfer function
L = turbulence scale term in the von Kármán transfer function
M = threshold value of the projection onto the tuned input
N = number of sample signals randomly generated
n = number of the best sample signals taken to form the sample mean
Pr = probability
p = probability value of the sample mean having specified properties
r_n = energy of the residual error at stage n
S = subspace of constant system response tangent to the tuned input at the energy shell
s_n = sample mean at stage n
U_n = energy of the base signal at stage n
$U(x)$ = energy of the signal x
U_σ = design envelope analysis peak gust intensity value
V = velocity of the aircraft in the von Kármán transfer function
X = gust input signal in the Fourier plane, $X(\omega)$
x = gust input signal in the time plane, $x(t)$
$\|x\|$ = norm of the signal x
x_E = residual error in the tuned input solution
x_T = tuned input for a given system and energy level
$\langle x, y\rangle$ = inner product of the signals x and y
Y = system response threshold
y_d = design load
β, γ, θ = angles
δt = time step value
σ_g = standard deviation of the gust inputs in the von Kármán transfer function
σ_n = standard deviation of the sample signals at stage n
σ_{proj} = standard deviation of the projection of the sample mean onto the subspace spanned by the tuned input
σ_x = standard deviation of the input process
σ_y = standard deviation of the output process
τ = time duration of an input signal
ϕ_g = power spectrum of the gust inputs
$\Psi(\omega)$ = frequency response function

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I. Introduction

A GREAT deal of research has been devoted to predicting the response of aircraft to atmospheric turbulence for the purposes of stress analysis[1,2] and design and certification.[3–9]

In stress analysis,[1,2] a study is made of the various response loads, referred to as time-correlated, or balanced, loads of a system when one particular load, known as the reference load, is maximal. When the system is linear, the values of each output can be calculated either by a cross-correlation technique[2] or by a technique based upon matched filter theory (MFT)[2]; the results of both methods being theoretically identical.[2] When the system is nonlinear, such techniques are no longer applicable. We therefore consider an existing generalization[5] of the MFT method that extends to cover nonlinear systems. Central to this extension is the calculation of a particular gust input time history that maximizes the reference load, subject to a constraint that is defined by an existing requirement.[8,9] This gust time history is then fed back into the system and the time histories of each output load are then calculated. The aim of this paper is to describe a stochastic search algorithm that performs a search for such gust input histories, referred to as "tuned inputs."

In the area of design and certification, in particular in the traditional design envelope approach to safety[8,9] and reviewed in Jones,[5] a response factor is calculated and multiplied by a given gust intensity to obtain a peak design load. The response factor is calculated using knowledge of the simple relationship between the statistics of the gust inputs and the statistics of the response loads, and is applicable only in the case of linear systems. This definition of design load is not applicable for nonlinear systems because the simple relationship between the statistics of the gust inputs and the response loads no longer holds, invalidating the definition of the response factor. A number of alternative definitions of design loads have been proposed and compared in the literature.[3–5,7,10,11] However, these definitions are only equivalent for linear systems. Although they do extend to cover nonlinear systems meaningfully, the design loads defined by each method differ both in value and meaning. We choose one of these definitions (by Jones[5]) where it was shown that the design loads for linear systems are produced by specific inputs, also called tuned inputs, which maximize the response load of the system when the gust inputs are subjected to an energy constraint related to the safety requirement. By defining the design load to be the response of the system to this tuned input, Jones's approach extends to nonlinear systems. In this paper we show that the proposed new stochastic method for estimating tuned inputs for use in calculating time-correlated loads in stress analysis may also by used for estimating tuned inputs for use in calculating design loads using Jones' definition. Note, however, that we do not attempt to show that the stochastic method is a computationally efficient way of calculating design loads if that is all that is required. The primary aim of the stochastic method is to estimate tuned inputs to calculate time-correlated, or balanced, loads for use in stress analysis.

In Scott et al.,[7] two methods are detailed in the design and certification problem: a deterministic method based upon the principles of MFT[12] and a stochastic method related to stochastic simulation. We show that both methods may be interpreted in the wider context of searching for tuned inputs, and as such are used as baseline methods for comparison with the stochastic search method proposed herein.

In Sec. II we formally define tuned inputs and the problem addressed by this paper. We also describe how tuned inputs can be used both in the study of time-correlated loads and in the problem of design load prediction. In Sec. III we describe both the matched filter based (MFB) method and the stochastic simulation based (SSB) method of Scott et al.,[7] and their relation to the search for tuned inputs. As a precursor to the development of a new stochastic search method, the incremental stochastic search (ISS) method of Sec. V, we first give an alternative interpretation of Scott's algorithm in Sec. IV. Finally, in Sec. VI, we demonstrate the performance of the ISS method on two configurations of a realistic aircraft model: a linear and a nonlinear configuration. With the linear configuration we show that the ISS method does indeed converge to the ideal tuned input by comparing the solution generated with the ideal tuned input produced using the results of MFT.[12] Comparisons are also given of the performance of the ISS and SSB methods. With the nonlinear configuration we compare the performance of the ISS method with the performance of the MFB and SSB methods of Scott et al.[7]

II. Problem Definition

In this section we first describe the space of input signals and the systems under consideration before defining tuned inputs. We then show how the definition applies to both the time-correlated loads and design and certification problems.

We assume that the space of all possible input signals $x(t)$ has the following inner product:

$$\langle x_1, x_2\rangle = \int_{-\infty}^{\infty} x_1(t)x_2(t)\,\mathrm{d}t \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are both input signals.

This inner product allows us to define an energy functional on the input space by

$$U(x) = \langle x, x\rangle = \|x\|^2 \tag{2}$$

Alternatively, energy can be defined in an equivalent manner in the frequency plane by

$$U(x) = \int_0^{\infty} |X(\omega)^2|\,\mathrm{d}\omega \tag{3}$$

where X is the Fourier transform of the time-dependent input $x(t)$.

We assume that the response of any given system acting on such input signals is given by a function F. In particular, the maximum response generated by the system when the input is $x(t)$ is given by the functional f defined by

$$f(x) = \max_t F[x(t)] \tag{4}$$

By combining the energy functional [Eq. (2)] and the maximum response functional [Eq. (4)], we can now formally state the problem addressed in this paper to be "Find the global maximum of f when $x(t)$ is restricted to the set of inputs that satisfy the energy constraint $U(x) = U$."

We call the input signal that generates this global maximum the *tuned input* of the system at that energy level.

In this paper, and with the exception of translation with respect to time, we assume that the systems have a unique tuned input for a given energy level. For some nonlinear systems this is not the case; however, the algorithms outlined in this paper may be augmented with cluster-recognition techniques, such as correlation checking, to deal with those systems that have more than one tuned input for a given energy level. Note that we also make the assumption that the maximum response of the system at a particular energy level is a monotonically increasing function of that energy level.

In the study of time-correlated loads, and for a given energy level, a tuned input for the specified system reference load can be estimated using the stochastic search method to be described herein, and then fed back into the system to study its effects on the remaining load outputs.

For linear systems the tuned input will have the same shape, irrespective of energy level, but for more general nonlinear systems the shape of the tuned input will change, sometimes discontinuously, with the energy level. Therefore, the nonlinear nature of the systems dictates that separate tuned inputs must be calculated for each energy level of interest.

In the standard design envelope approach to limit loads for both design and certification purposes, the design load y_d of a linear aircraft system is calculated using the relationship[5,8,9]

$$y_d = \bar{A}_y U_\sigma \tag{5}$$

where U_σ is a turbulence intensity prescribed in the safety requirements, and $\bar{A}_y$ is a system-dependent response factor calculated as the ratio of the standard deviations of system output y and gust velocity input g:

$$\bar{A}_y = \sigma_y/\sigma_g \tag{6}$$

Because the aircraft system is linear, $\bar{A}_y$ can be calculated in the frequency domain using the equation

$$\sigma_y^2 = \int_0^{\infty} |H(i\omega)|^2\,\phi_g(\omega)\,\mathrm{d}\omega \tag{7}$$

where H is the frequency-response function of the system subject to a stationary, stochastic gust velocity input g whose power spectrum ϕ_g incorporates σ_g.

For nonlinear systems, there is no equivalent to Eq. (7), and the meaning of $\bar{A}_y$ becomes unclear. However, Jones[5] showed that the design load y_d for a linear system is equal to the maximum load generated by the system when the deterministic inputs $x(t)$ have their energies $U(x)$ constrained by

$$U(x) \le U_\sigma^2 \tag{8}$$

In other words, the design load

$$y_d = \max_{U(x)\le U_\sigma^2} |y(x)| \tag{9}$$

is the response of the system to the tuned input of energy level U_σ^2. In the preceding argument, the input gusts were assumed to have uncorrelated (white) Gaussian statistics; however, the input processes with the more general power spectral density $|\Psi(i\omega)|^2$ may be dealt with by augmenting the original system with a linear prefilter whose frequency response function is $\Psi(i\omega)$. Note that following Scott et al.,[7] we also may choose to take the energy in the time-correlated loads problem to be U_σ.

Because this equivalent determination of the design load contains no reference to linearity, Jones[5] proposed that Eqs. (3) and (9) be used to calculate design loads for both linear and nonlinear systems. For linear systems, the results would be equivalent to those generated using Eq. (7).

III. Existing Search Methods

In this section we detail two existing methods, originally proposed for determining the design loads in the design and certification problem, which we show can be interpreted as searching for tuned inputs.

Scott et al.[7] introduced two methods for determining the design loads of systems: a deterministic MFB method and a SSB method.

A. MFB Method

Consider first the MFB method.[7] In the transfer function approximation of the von Kármán spectral density function

$$H(\omega) = \sigma_g \sqrt{\frac{L}{\pi V} \frac{1 + \frac{8}{3}[1.339 i\omega(L/V)]}{[1 + 1.339 i\omega(L/V)]^{\frac{11}{6}}}} \tag{10}$$

where L is the scale of the turbulence and V is the velocity of the aircraft, the value of σ_g is taken to be U_σ. By augmenting the aircraft system with this transfer function we can create a combined system with inputs having the characteristics of Gaussian white noise. The MFB algorithm has the following steps (see Fig. 1):

1) A range of real values for the parameter k is chosen.

2) For each value k_i in the range of k, an impulse of strength k_i is generated and then fed into the combined system. The time-history output is then stored.

3) Each time-history output is reversed in time, normalized to have unit energy, and then fed back into the combined system, with the highest peak y_i of the new response being noted.

4) The largest of the peak responses, $\max_i(y_i)$, is then taken to be the design load.

Note that the same peak response would be generated if, instead of using $\sigma_g = U_\sigma$ and normalizing to unit energy, we used $\sigma_g = 1$ and normalized to energy U_σ^2.

The design load given by the MFB method is thus defined to be the largest peak response generated by the preceding algorithm. However, as also suggested by Scott et al.,[7] we could also consider the MFB algorithm to be conducting a search for a tuned input. We therefore take the MFB method to be a baseline method for searching for tuned inputs.

B. SSB Method

We next consider the SSB method.[7] With this method, the value of σ_g in the von Kármán filter is taken to be $(1/\eta)U_\sigma$, for some $\eta > 1$. The actual value of η is important and will be discussed in more detail following the description of the SSB algorithm.

In the SSB method (see Fig. 2) a time history of a stationary zero-mean uncorrelated Gaussian random process with unit standard deviation $\sigma_x = 1$ is generated and fed into the combined system with the time-history output being recorded. (Note that instead of taking $\sigma_g = (1/\eta)U_\sigma$ and $\sigma_x = 1$, we could also have chosen $\sigma_g = 1$ and $\sigma_x = (1/\eta)U_\sigma$.) The peak values of the time-history response are noted and those with the highest magnitude within a time interval of $\pm\tau$ seconds are identified. The time-history inputs and outputs within these identified intervals are extracted and aligned so that the peak responses occur at the same time, and they are then averaged to form both an average tuned input and an average tuned output.

Scott et al.[7] suggest that the average tuned output peak response is a reasonably accurate estimate of the peak design load y_d for a linear system [Eq. (6)], when η is $\sim$3.

Alternative definitions for design loads that have also been implemented using a stochastic simulation process similar to the preceding method suggest different values and methods for determining a good value of η. For instance, in Gould,[10] the value for η is determined by performing stochastic simulations over a number of different values of σ_g. For each value of σ_g, a stochastic simulation run takes place, and $\eta(\sigma_g)$ is assigned to be the ratio of the peak response to the rms of the response, $\eta(\sigma_g) = y_{\max}(\sigma_g)/y_{\mathrm{rms}}(\sigma_g)$. A graph is then plotted of $U(\sigma_g) = \eta(\sigma_g)\sigma_g$ against $y_{\max}(\sigma_g)$, and the design load is estimated using interpolation at the value $U(\sigma_g) = U_\sigma$. It is important to note, however, that the latter method has no obvious interpretation as solving the problem defined by Eqs. (3) and (9),

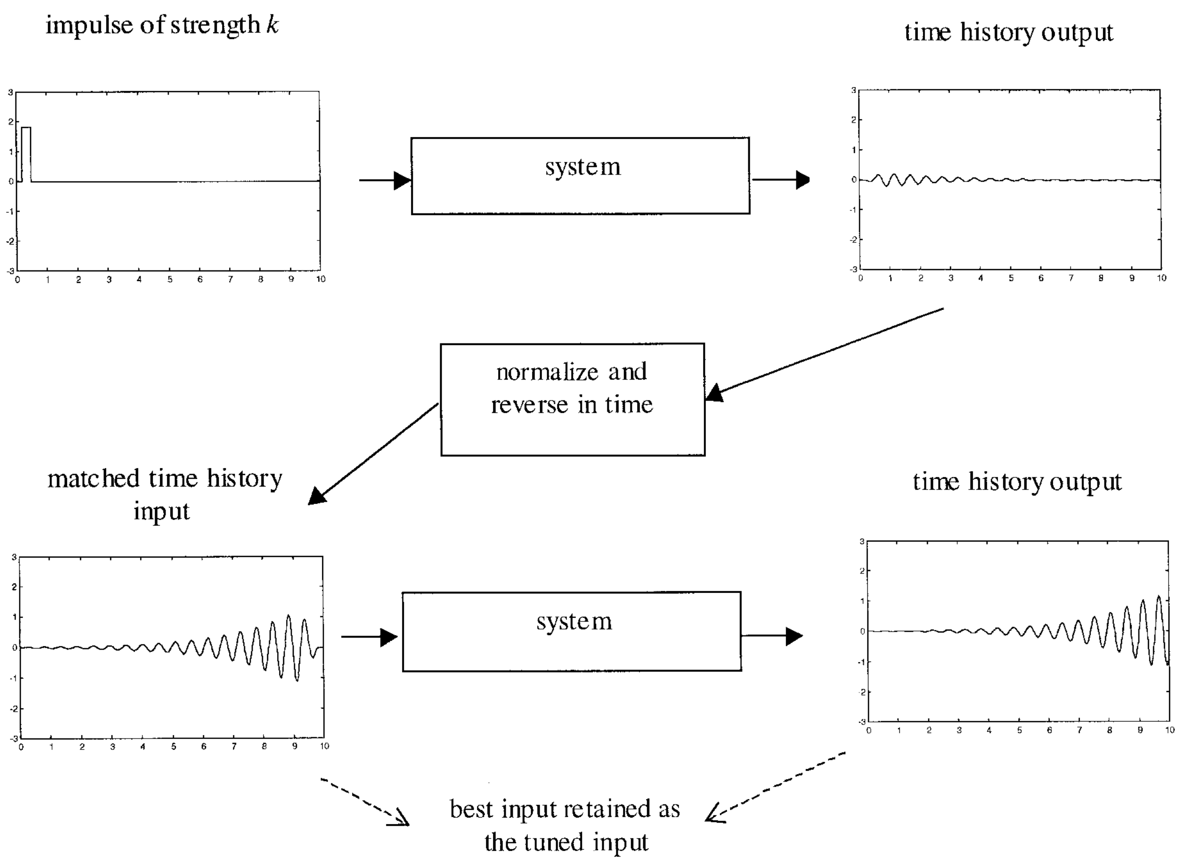

Fig. 1 MFB search method for tuned input searching.

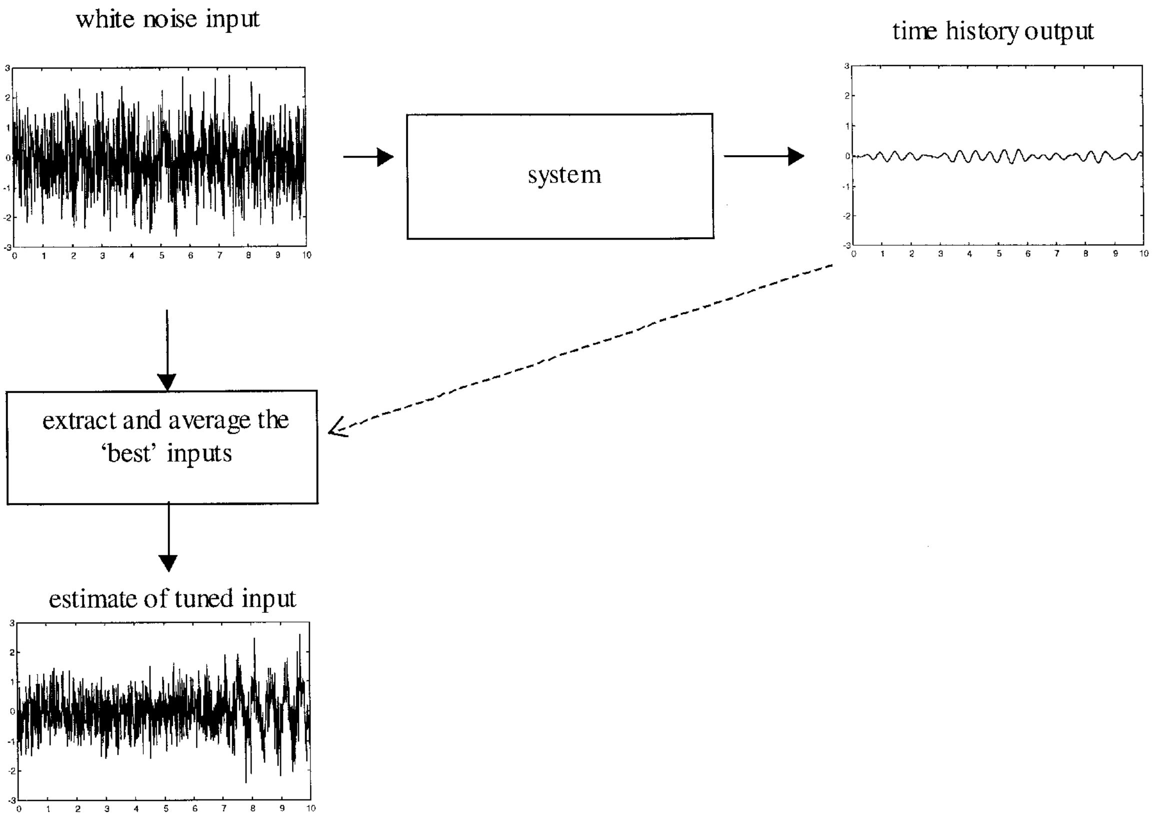

Fig. 2 SSB method adapted for tuned input searching.

but serves instead as a different definition of design load. Indeed, no tuned input is produced.

However, in the next section, we show how a value for η used in the SSB method can be calculated explicitly for the SSB method when it is interpreted as generating tuned inputs and outputs.

IV. New Interpretation of the SSB Method

As a precursor to the ISS method to be introduced in Sec. V, we now give a new interpretation of Scott et al.'s[7] SSB method. This new interpretation allows us to calculate an exact value for the term η, so that the expected tuned input solution matches the ideal tuned input exactly for the linear case, and approximately for nonlinear systems subject to certain conditions.

Note that we choose, without loss of generality, to take $\sigma_g = 1$ and aim to calculate an appropriate value for σ_x.

Let x_T be the tuned input of length τ seconds for the system f subject to the energy constraint $U(x_T) = U$. We make three assumptions about the system:

A1) The system response varies smoothly with respect to small changes in the input signal.

A2) There is a high degree of correlation between the peak system response produced by an input x and the projection $\langle x, x_T\rangle$ of the input onto the subspace spanned by the tuned input x_T. The justification for this assumption is that input patterns that excite the system are very likely also to excite the system by a similar amount when they are embedded in noise, particularly if the pattern and the noise are not cross correlated.

A3) The set of inputs with high peak system response, whose energy is less than the prescribed energy constraint U, is approximately convex. This ensures that the conditioned averaging process produces a solution with reduced error without significantly reducing its peak system response.

Assumption A2 is the key to the effectiveness of the stochastic search methods described in this paper because it greatly reduces the scale of the search. The probability of generating inputs containing both the tuned input and a noise component is far larger than the probability of generating inputs that have no noise. The method is therefore designed to search for the former, using the averaging process as a way of canceling the noise and thus reducing the error in the final solution.

Note that the ISS algorithm proposed in Sec. V of this paper will converge to the tuned input of a system satisfying the preceding assumptions; however, it is anticipated that the proposed ISS algorithm will also produce good results for systems where the assumptions are relaxed somewhat.

Central to our interpretation of Scott et al.'s[7] SSB method is the idea of the projection of the system input $x = x(t)$ onto the subspace spanned by the input $g = g(t)$. Such a projection produces the input

$$(\langle x, g\rangle / \langle g, g\rangle)g \tag{11}$$

which has energy

$$\left\langle \frac{\langle x, g\rangle}{\langle g, g\rangle}g, \frac{\langle x, g\rangle}{\langle g, g\rangle}g \right\rangle = \frac{\langle x, g\rangle^2}{\langle g, g\rangle} = \left\langle x, \frac{g}{\|g\|}\right\rangle^2 \tag{12}$$

Suppose that x is the input formed by averaging those n inputs of length τ seconds whose system response exceeds the threshold Y. We seek to choose Y so that the projection of the sample mean x onto the ideal tuned input x_T would be

$$E(\langle x, x_T\rangle \mid f(x) > Y) = \sqrt{U} \tag{13}$$

This ensures that the sample mean x contains the ideal tuned input at the correct energy level together with a residual error, i.e.,

$$x = \sqrt{U}\frac{x_T}{\|x_T\|} + x_E$$

where x_E is the residual error term that is orthogonal to the tuned input as shown in Fig. 3.

Using assumption A2, we suppose that choosing those inputs whose system response exceeds the threshold Y is equivalent to choosing those inputs whose projection $\langle x, x_T\rangle$ onto the tuned input

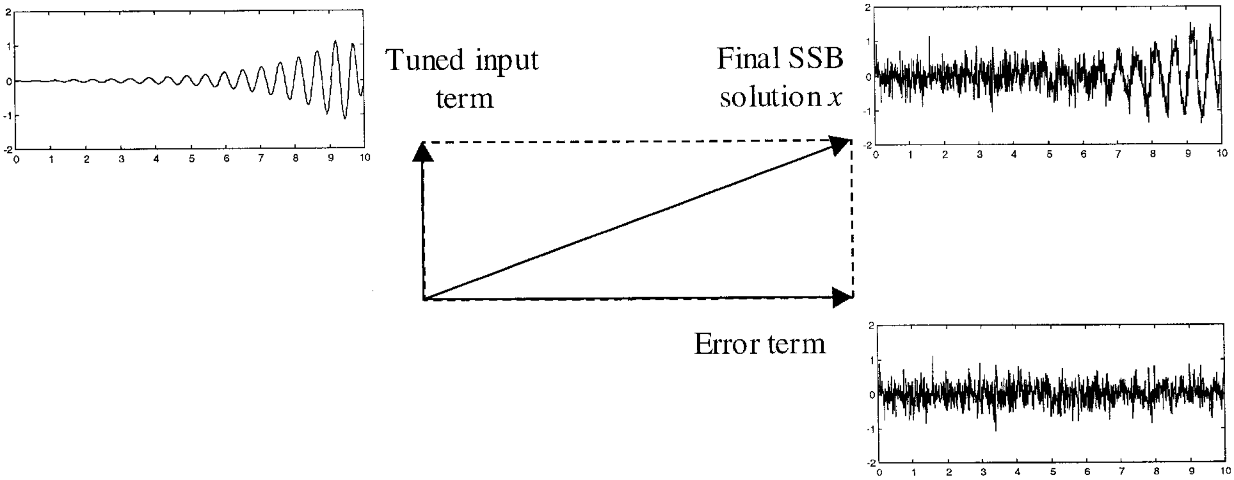

Fig. 3 Error in the SSB final solution.

exceeds an equivalent threshold M. If $M = k\sigma_{\text{proj}}$ is some scalar multiple of the standard deviation of the projection, then the constraint in Eq. (13) becomes

$$E[\langle x, x_T\rangle \mid f(x) > Y] = E(\langle x, x_T\rangle \mid \langle x, x_T\rangle > k\sigma_{\text{proj}}) = \sqrt{U} \tag{14}$$

We now consider the process of generating N random inputs of length τ seconds. There are two possibilities: we could either produce N separate instances of a random process of length τ [a total of $N(\tau/\delta t)$ samples, where δt is the duration between two samples], or we could generate one input of length $(N+1) + (\tau/\delta t)$ samples. Both methods will provide random inputs with the same desired properties, because the random process is uncorrelated, but the latter method, similar to the long-duration time-history inputs used in the SSB method, is computationally more efficient.

So that, out of a total of N inputs generated, the sample mean formed from those n satisfying $\langle x, x_T\rangle > k\sigma_{\text{proj}}$ satisfies Eq. (14), we should have the probability p of such inputs occurring to be n/N.

Because the input samples are Gaussian and uncorrelated, the projection is also Gaussian and has a standard deviation $\sigma_x\sqrt{U}$. The probability of generating an input signal whose projection onto the tuned input exceeds $k\sigma_{\text{proj}}$ is therefore,

$$\begin{aligned} Pr(\langle x, x_T\rangle > k\sigma_{\text{proj}}) &= Pr\left(\langle x, x_T\rangle > k\sigma_x\sqrt{U}\right) \\ &= Pr\left[\sqrt{U}\langle x, (x_T/\|x_T\|)\rangle\right] > k\sigma_x\sqrt{U} \\ &= \tfrac{1}{2}\left[1 - \text{erf}\left(k/\sqrt{2}\right)\right] \end{aligned} \tag{15}$$

where erf is the error function.[13] By equating Eq. (15) with p we can then solve for k.

Similarly, Eq. (14) can be also written as

$$E(\langle x, x_T\rangle \mid \langle x, x_T\rangle \geq k\sigma_{\text{proj}}) = \left(1/p\sqrt{2\pi}\right)\sigma_x \exp[-(k^2/2)] \tag{16}$$

If we now substitute the newly found value for k into Eq. (16) and equate to $\sqrt{U}$, we can then solve for σ_x.

Thus, given a probability $p = n/N$ and an energy level U, we can calculate k and then σ_x, which completely determines the properties of the input random process.

Following the SSB method,[7] we proceed to generate N instances of a zero-mean, uncorrelated, stationary Gaussian random process with standard deviation σ_x, and average those n with the highest system response to form a tuned input approximation. Whereas Scott et al.[7] take the standard deviation of the input process to be $(1/\eta)\sqrt{U} = \frac{1}{3}\sqrt{U}$, we choose

$$\sigma_x = p\sqrt{2\pi}\exp(k^2/2)\sqrt{U} \tag{17}$$

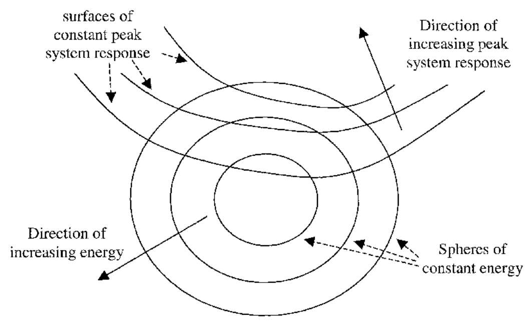

Fig. 4 Representation of the search space.

In other words, we take $\eta = (1/p\sqrt{2\pi})\exp[-(k^2/2)]$. Note that this value of η is only suitable for the SSB method when it is applied to the problem of finding a tuned input and not to other stochastic simulation methods that aim to generate estimates of design loads.

The choice of threshold exceedance probability p involves a tradeoff between frequently generating solutions with large error and infrequently generating solutions with small error. When a large value is taken for p, inputs containing patterns with sufficient energy to excite the system will be found in large numbers, but they will also contain large error terms. If the errors are uncorrelated, then the averaging process will reduce the error in the final solution; however, the average will need to be taken of a large number of samples to reduce the error significantly. If, on the other hand, a small value is taken for p, inputs containing patterns with sufficient energy to excite the system will generally have small errors, but will be generated much more rarely, requiring a large set of inputs, and therefore, a large number of system response evaluations.

V. Incremental Stochastic Search Method

The stochastic search method described in the previous section produces solutions with relatively large residual errors. In other words, the energy U taken up by the ideal tuned input part of the solution $x = \sqrt{(U)}x_T + x_E$ is much less than the energy of the residual error x_E in the final solution (see Fig. 3).

We therefore describe a new stochastic search method, ISS, which progressively reduces this error over a series of iterations. Consider Fig. 4, which is a two-dimensional representation of the search space where each point in the space is a signal of length τ seconds. There are two important surfaces shown: 1) surfaces of constant energy, which are the boundaries of hyperspheres of radius equal to the energy; and 2) surfaces of constant peak system response, which for linear systems will be hyperplanes, but which for nonlinear systems in general may have any shape. The surface of constant system response for the energy constraint U will be locally tangent to the energy shell $\{x:U(x)=U\}$ at the tuned input x_T, and it will be called the tangent surface S. Note that assumptions A1–A3, made about the nature of the system's nonlinearity, ensure that the tangent surface is locally smooth, and therefore, approximately flat near the tuned input.

We now describe the first stage of the proposed ISS algorithm.

Suppose we are given U. We decide on the number N of system inputs of length τ to randomly generate and evaluate, and the number n of those that cause the greatest peak response, and of which an average is sought. These three values allow us to calculate p, k, and then, via Eq. (17), the standard deviation σ_x of the random process used to generate the inputs. We then perform the stochastic search method described in Sec. IV, which should generate a final tuned input solution b_1 lying near the tangent surface, but which is typically distant from the tuned input in energy terms. The aim of the remainder of the ISS method is to converge along the tangent surface S toward the ideal tuned input.

At each of the next stages, a biased stochastic search centered on the current solution b_n and constrained to the energy shell $\{x:U(x)=U_n=U(b_n)\}$ is performed. The new solution formed at each stage by averaging the best inputs is then reduced in overall energy so that over a series of stages the energy level of the solution decreases toward the prescribed energy constraint. This process is conducted in such a manner to maintain the energy of the ideal tuned input part of the solution at U while reducing the energy of the residual error, as will be described next.

With reference to Figs. 5 and 6, we propose the following algorithm:

1) Create N instances $\{x_i=x_i(t)\}_{i=1}^{N}$ of a zero mean, uncorrelated (white) Gaussian random process of length τ seconds and standard deviation σ_n.

2) Add each instance x_i to the base signal b_n to produce N new input signals $\{b_n+x_i\}_{i=1}^{N}$.

3) Normalize the energy of each new input signal so that it lies on the energy shell $\{x:U(x)=U_n\}$.

4) Feed each normalized input into the system and take those n signals with the highest system responses and average them to form a sample mean s_n.

5) Because the new signals were formed by adding uncorrelated noise to the base signal, the sample mean x_m will be formed by those signals with an increased amount of ideal tuned input energy, i.e., $x_m=[\surd(U)+\varepsilon](x_T/\|x_T\|)+x_E$ for some $\varepsilon>0$. In other words, the sample mean is no longer on the tangent space S for the desired energy level U, but has migrated to a similar tangent space for the higher energy level $[\surd(U)+\varepsilon]^2$. To retain the base signal on the desired tangent space S, we must therefore reduce the energy of the sample mean to a new value U_{n+1} lying in the open interval (U, U_n) by multiplying the sample mean by $\surd U/[\surd(U)+\varepsilon]$. The energy adjusted sample mean becomes the base signal b_n for the next stage.

Because the system is assumed to be locally linear by assumption A1, points with relatively high system response on the energy shell $\{x:U(x)=U_n\}$ will also produce relative high system response when they are reduced in energy onto the shell $\{x:U(x)=U_{n+1}\}$, as long as $|U_n-U_{n+1}|<\delta$ for some sufficiently small $\delta>0$. Thus, as long as we reduce the energy level slowly, the algorithm will converge to the tuned input along the subspace S.

Again consider Fig. 6. At the nth stage, there are two parameter values we need to determine for the next stage: the energy constraint U_{n+1} and the standard deviation σ_{n+1} of the random process.

As we seek to reduce the energy slowly, we stipulate that the energy $r_n=\surd(U_n^2-U^2)$ of the expected error at each stage be reduced by cr_n, where $0<c<1$ is generally much smaller than 1. The next energy constraint for the $(n+1)$th stage should therefore be $U_{n+1}=\surd[U^2+(1-c)^2r_n^2]$.

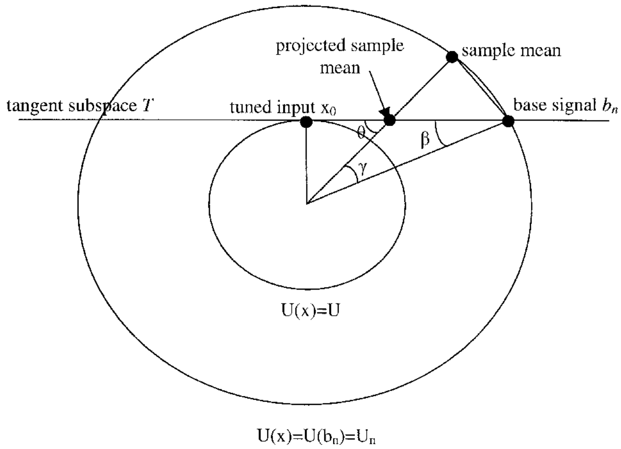

Fig. 6 Representation of an ISS stage.

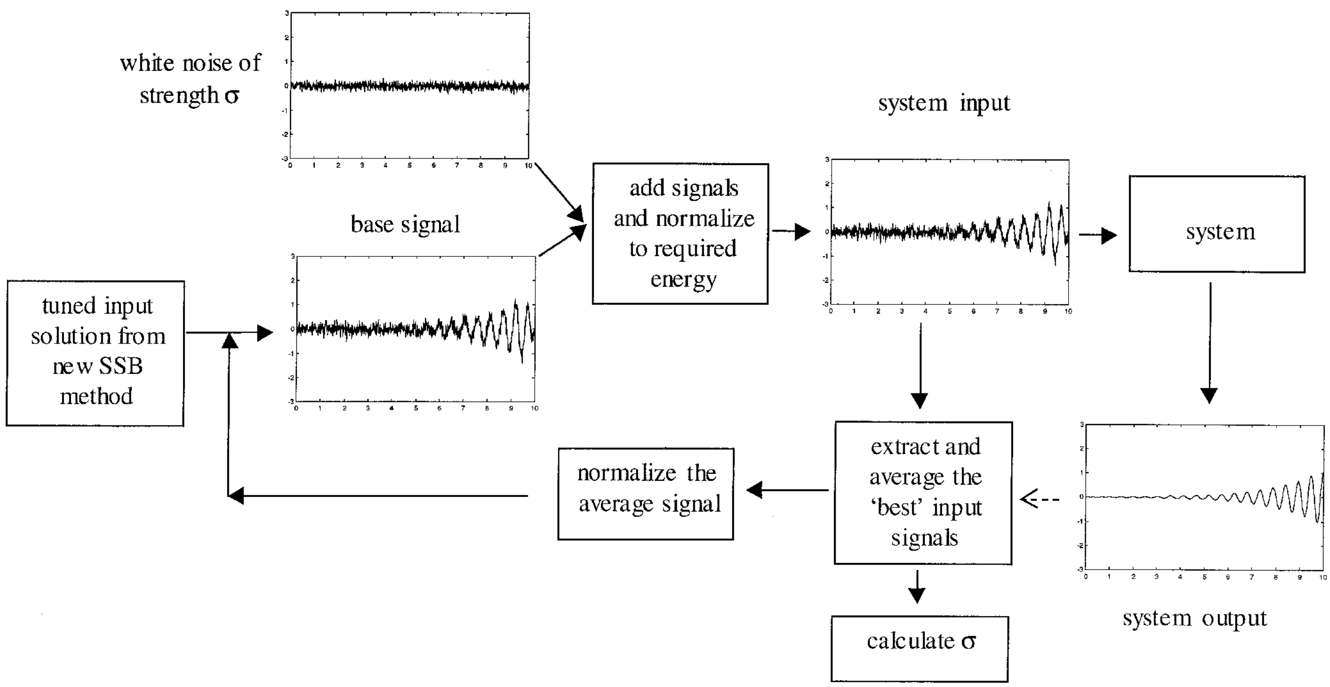

Fig. 5 ISS method for tuned input searching.

To calculate the standard deviation σ_{n+1} for the $(n+1)$th stage, we first note that $\theta = \tan^{-1}[U/(1-c)r_n]$, $\beta = \sin^{-1}(U/U_n)$, and $\gamma = \theta - \beta$. From this we estimate the energy difference d_{n+1} between the expected sample mean S_{n+1} and the current base signal b_n to be

$$d_{n+1} = \sqrt{2U_n^2(1-\cos\gamma)} \tag{18}$$

Now, for standard deviation σ_{n+1}, the expected value of d_{n+1} is also given by

$$d_{n+1} = \left\{\frac{\sigma_{n+1}\exp[-(k^2/2)]}{p\sqrt{2\pi}}\right\}^2 \tag{19}$$

We may therefore equate Eqs. (18) and (19) to calculate σ_{n+1}. We obtain

$$\sigma_{n+1} = p\sqrt{2\pi}\exp(k^2/2)\sqrt[4]{2U_n^2(1-\cos\gamma)} \tag{20}$$

Note that as we converge toward the tuned input, the standard deviation of the stages decreases toward zero. This ensures that the perturbations we add gradually decrease in energy, ensuring that the convergence is stable.

Thus, at each iteration of the algorithm, we can calculate the values of the energy constraint U_{n+1} and the standard deviation σ_{n+1} to use at the next iteration to ensure the convergence along the subspace S toward the ideal tuned input of energy U.

VI. Performance Tests

We now demonstrate the performance of the new ISS method on two configurations of a dynamic aircraft model. With the linear configuration, the convergence of the ISS solution toward the ideal tuned input generated via the MFT (or equivalently, the MFB method) is demonstrated. With the nonlinear configuration, the performance of the ISS method is compared with the performance of both the MFB and SSB methods.

The system we consider for performance tests is a realistic model of a structural component from a commercial wide-bodied transport aircraft.[6] A schematic of the system is given in Fig. 7, whereas full details of the aircraft system and the equations of motion may be found in the report.[6]

The input to the system is a wind-gust-velocity time history having the following modified von Kármán spectrum[6]:

$$|H(i\omega)|^2 = \sigma_g^2 \times 0.9162 \times \frac{1+39.6125\omega^2}{(1+14.8547\omega^2)^{\frac{10}{6}}} \tag{21}$$

A causal preshaping filter having the frequency response function

$$H(i\omega) = \sigma_g\sqrt{0.9162} \times \frac{1+\sqrt{39.6125}i\omega}{\left(1+\sqrt{14.8547}i\omega\right)^{\frac{10}{6}}} \tag{22}$$

is therefore applied to the Gaussian white inputs before they are fed into the system.

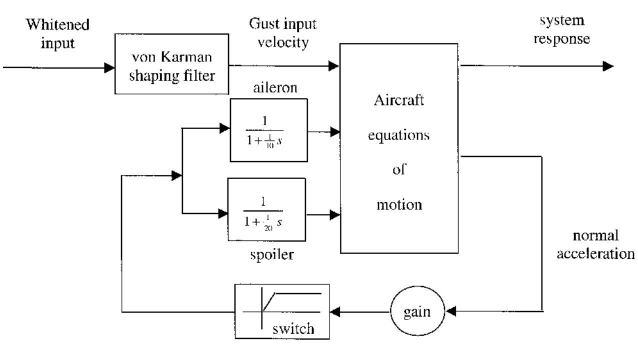

Fig. 7 Schematic of the aircraft subsystem model.

Each input signal is of length $\tau = 10$ s. The signal consists of 1024 time steps of duration $\delta t = \frac{10}{1024}$ s. The energy constraint on the input space is taken to be $U = 1$ with the standard deviation of the gust input in the von Kármán transfer function, Eq. (22), taken to be $\sigma_g = 85$.

We consider two system outputs: the engine's lateral acceleration and the normal acceleration of the aircraft's c.g., the latter being fed back through the system via a switch.

In the linear configuration, the switch is permanently open so that no feedback is transmitted. The system output is the engine's lateral acceleration.

In the nonlinear configuration, the switch is initially open, but when the normal acceleration exceeds a threshold value of $0.15g$, the switch becomes permanently closed, allowing uninterrupted feedback. The system output is the normal acceleration of the aircraft's c.g.

For the MFB, SSB, and ISS methods, 30,000 function evaluations were used, where each function evaluation produces a complete time-history output of length $\tau = 10$ s (or 1024 time steps).

For the MFB method, 15,000 values of k were chosen uniformly in the range of 1–10,000. Therefore, 15,000 function evaluations were required for evaluating the system response of the impulses of strength k, with a further 15,000 required to calculate the response of the system to the time-reversed outputs produced by these impulses.

For the SSB method, one time-history input of length 3×10^5 s was used with the best 20 intervals of a length of 10 s taken to form the average according to the SSB requirements.

For the ISS method, 30 stages of the algorithm were specified, each using 1000 function evaluations and taking the 20 best inputs to form the average. The first stage involved using the SSB method with a time-history input of length 1×10^4 s with the 20 best intervals of a length of 10 s taken to form the sample mean. Each of the remaining stages consisted of 1000 time histories, each a length of 10 s being evaluated, with the 20 best inputs being used to form the sample mean, and hence, the base signal for the next stage.

For each system, the SSB and ISS algorithms were run 10 times to take into account the stochastic nature of the algorithms and improve the reliability of the comparison.

A. Linear Aircraft Configuration

In the aircraft model's linear configuration, the switch is permanently open and the feedback loop is therefore deactivated. The system output is taken to be lateral acceleration of the engine.

Because the system is linear, the tuned input can be calculated exactly using the results of MFT, or equivalently using the MFB method, and therefore can be used as a base for comparison. The ideal tuned input, shown in Fig. 8, generates a peak system response of $1.1813g$.

The tuned input solutions generated by the SSB method gave peak system response values in the range of 1.1594–$1.2080g$. However, once normalized to have unit energy, these tuned inputs gave peak system responses in the reduced range of 0.4905–$0.5421g$, indicating the large effect of the error when using the definition of

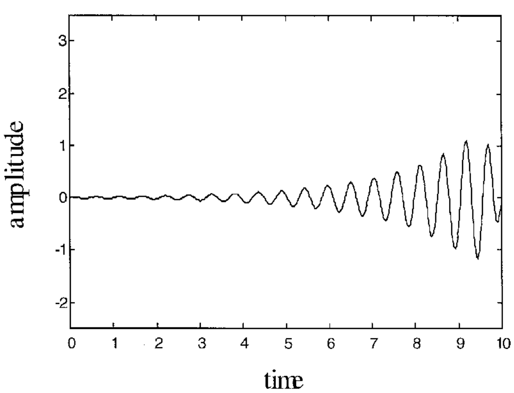

Fig. 8 MFB linear aircraft solution.

the design load given by Eq. (9). Direct calculation of the ideal tuned input part of the best SSB solution, shown in Fig. 9, reveals it to have energy 0.9898 with an orthogonal error of energy 4.5918. The ratio of the energy of the error to the energy of the ideal tuned input part of the solution is therefore about 4.5.

The tuned input solutions produced by the ISS method, once normalized to unit energy, generated peak system responses in the range of 1.0352–1.0722g, roughly 90% of the ideal system response. The ideal tuned input part of the best ISS solution, shown in Fig. 10, has energy 0.9151, with the energy of the orthogonal error now 0.4481. The ratio of the energy to the error ideal tuned input energy is therefore roughly $\frac{1}{2}$. Comparison with the SSB method ratio of 4.5 clearly shows how much cleaner the ISS solution is. In Figs. 11 and 12, we indicate how the energy of the ISS base signal and its associated system response converge over 30 stages for all 10 runs of the algorithm. In Fig. 13, we show that although the energy of the base signal is reducing toward the energy constraint, the energy of the ideal tuned input part of the base signal remains fairly constant. This indicates that it is the energy of the error that is being reduced over the algorithm's stages.

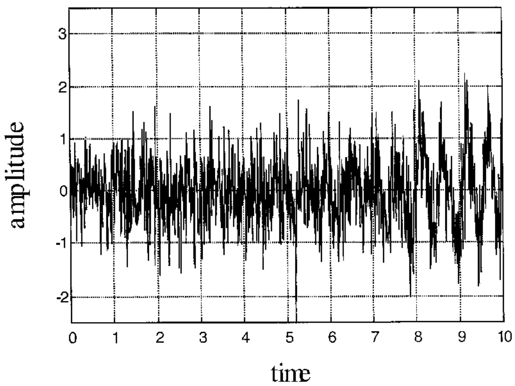

Fig. 9 SSB linear aircraft solution.

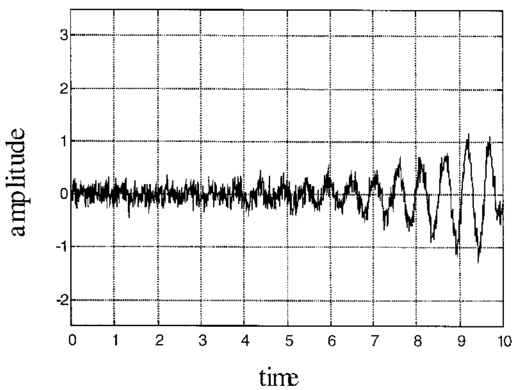

Fig. 10 ISS linear aircraft solution.

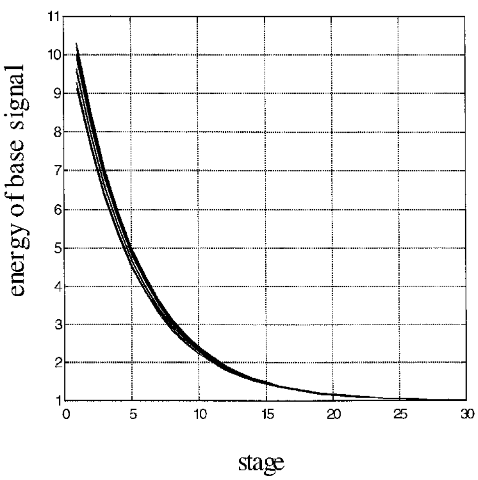

Fig. 11 ISS energy reduction for linear system.

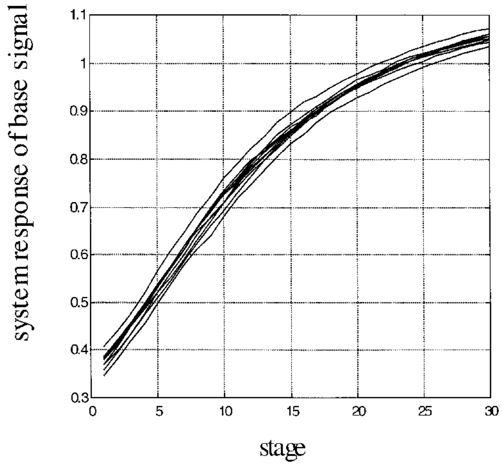

Fig. 12 ISS convergence of system response for linear system.

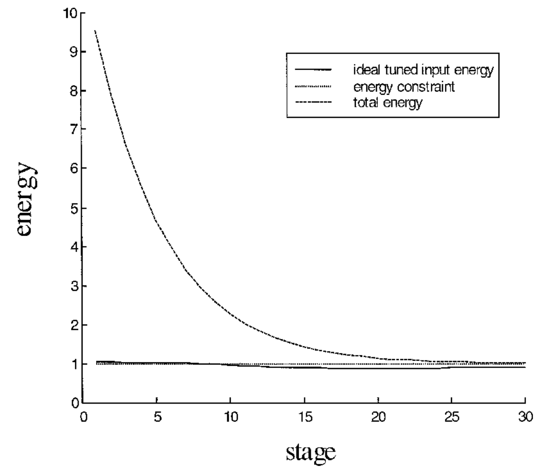

Fig. 13 ISS linear aircraft comparison of total energy of solution and energy of the tuned input part of the solution.

B. Nonlinear Aircraft Configuration

In the aircraft model's nonlinear configuration, the normal acceleration of the aircraft's c.g. is chosen to be the system output, and the feedback loop becomes activated when the value of this normal acceleration exceeds a particular threshold (0.15g). Once activated, the switch remains permanently closed.

In Figs. 14–16 we show the best tuned input solutions produced by the MFB, SSB, and ISS algorithms. In this case, because we no longer know the ideal tuned input, and because superposition no longer holds, we therefore compare the quality of the solutions using the definition of design load given by Eq. (9). The system assumptions made throughout the paper ensure that those inputs with the highest system response, when normalized to a specific energy level, are the best solutions.

The MFB solution appears to have a smooth and clean signal; however, the system response generated by the solution reached only 0.6413g. The system responses generated by the best ISS solutions lay in the higher range of 0.7313–0.8563g, an improvement over the MFB solution of between 14 and 34%. By comparison, the solutions generated by the SSB method, after normalization to

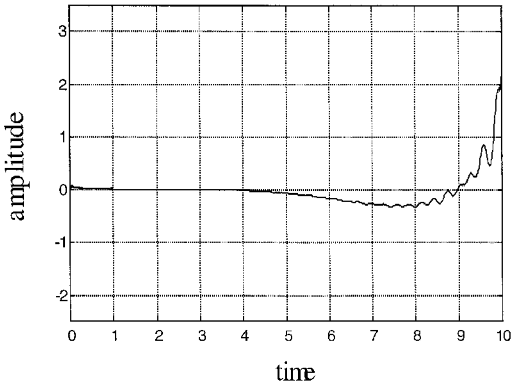

Fig. 14 MFB nonlinear aircraft solution.

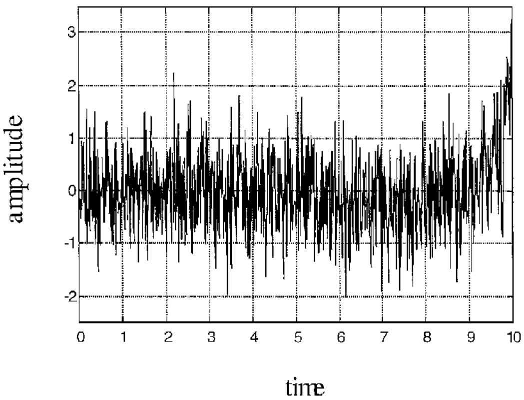

Fig. 15 SSB nonlinear aircraft solution.

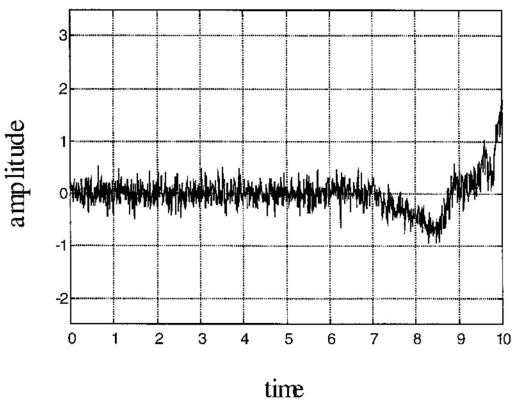

Fig. 16 ISS nonlinear aircraft solution.

unit energy to comply with the design load of Eq. (9), only produced peak system responses in the range of 0.1983–0.2379g. The difference in system responses of the ISS and MFB methods, although both solutions have similar energy levels, shows that although the MFB solution appears to be smoother and cleaner than the ISS solution, it has converged to a local rather than a global optimum.

In Fig. 17, we show how the energy of the ISS base signals decreases over stages. The reduction is smooth by definition, although the convergence of system response of the base signals in Fig. 18 is less so initially. It is believed that the initial slow convergence of some of the ISS runs is due to the initial base signal produced by the SSB method being far from the tangent surface; however, it is encouraging to note that even in this situation, the ISS solutions still converge toward the ideal tuned input.

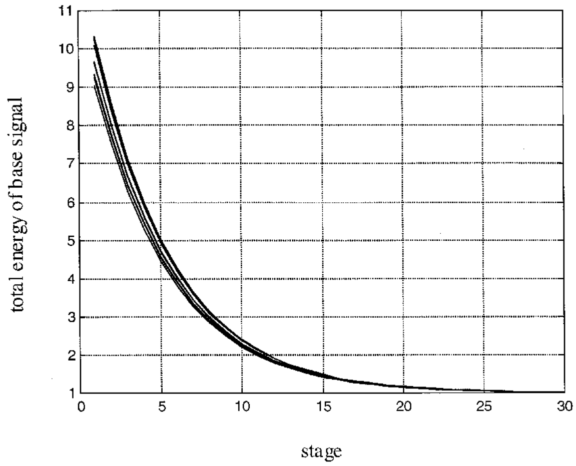

Fig. 17 ISS energy reduction over stages for the nonlinear aircraft.

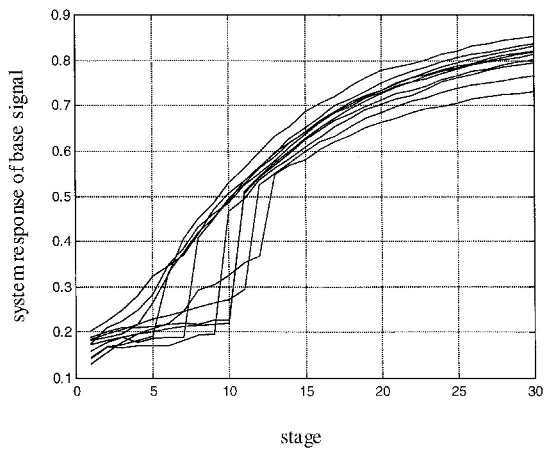

Fig. 18 ISS convergence of system response for the nonlinear aircraft.

VII. Conclusions

We have demonstrated a new stochastic method for the estimation of energy-constrained tuned inputs to linear and nonlinear systems, ISS, which has applications in the areas of stress analysis and design and certification of aircraft. The method has been demonstrated on models of two aircraft systems: one linear, where the tuned input is known exactly from MFT, and a nonlinear system. With the linear system, the ISS method was shown to converge to the ideal tuned input as predicted by the MFT and to produce tuned input solutions with much less error than the existing SSB method.[7] With the nonlinear system the method was compared with two existing alternative search methods developed by Scott et al.,[7] and shown to generate a tuned input solution with greater accuracy than either of the latter methods for the same computational expense.

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